Continuity and Differentiability

1. $f(x) = \dfrac{x^2 - 25}{x - 5}$ and $g(x) = x + 5$ are not identical because f(5) is undefined but g(5) = 10 is well-defined.

Since $\lim\limits_{x\to5} f(x) = \lim\limits_{x\to5} \dfrac{x^2-25}{x-5} = \lim\limits_{x\to5}(x+5) = 10$, f(x) should be redefined as $f(x) = \begin{cases} \dfrac{x^2-25}{x-5} & , x \neq 5 \\ 10 & , x = 5 \end{cases}$

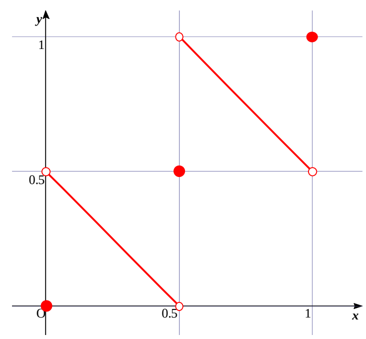
to remove the discontinuity.

2. Let $f(x) = \dfrac{5}{x-1} + \dfrac{7}{x-2} + \dfrac{16}{x-3}$, $\lim\limits_{x\to1^+} f(x) = +\infty$, $\lim\limits_{x\to2^-} f(x) = -\infty$, $\lim\limits_{x\to2^+} f(x) = +\infty$, $\lim\limits_{x\to3^-} f(x) = -\infty$

There are sign changes as x goes from 1^+ to 2^- and from 2^+ to 3^-. Also, x is continuous in (1, 2) and (2, 3), therefore f(x) = 0 has a solution between 1 and 2 , and another between 2 and 3 .

3. $\lim\limits_{x\to k^-} f(x) = \lim\limits_{x\to k^-}\left(\dfrac{x^2}{k} - k\right) = \dfrac{k^2}{k} - k = 0$, $\lim\limits_{x\to k^+} f(x) = \lim\limits_{x\to k^+}\left(k - \dfrac{k^3}{x^2}\right) = k - \dfrac{k^3}{k^2} = 0$, and f(k) = 0,

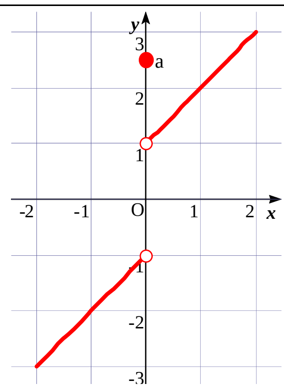
therefore f(x) is continuous at x = k .



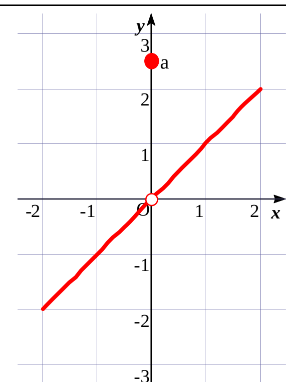
4. The graph on the right shows that f(x) assumes every value between 0 and 1 once and only once as x increases from 0 to 1 .
Also, the graph of y = f(x) and $y = f^{-1}(x)$ are identical.
The discontinuities of f(x) are x = 0, x = 0.5 and x = 1.

5.

(a) f(x) is not continuous and not differentiable at x = 0 for any value of a.



(b) f(x) is not continuous and not differentiable at x = 0 for any value of $a \neq 0$.
It is continuous and differentiable at x = 0 if a = 0.



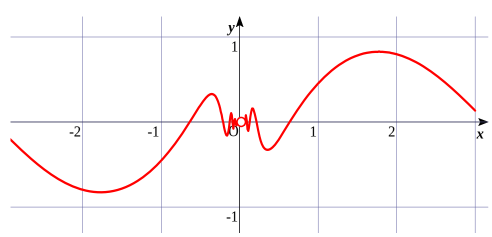
6. Since $-1 < \cos\dfrac{1}{x} < 1 \Rightarrow -\sin x < \sin x \cos\dfrac{1}{x} < \sin x$ for $x > 0$ and $-\sin x > \sin x \cos\dfrac{1}{x} > \sin x$ for $x < 0$.

Using Sandwich theorem, we get $\lim\limits_{x\to0} f(x) = \lim\limits_{x\to0} \sin x \cos\dfrac{1}{x} = 0$.

Also, f(0) = 0.

$\therefore$ f(x) is continuous at x = 0.

The graph of $g(x) = \sin x \cos\dfrac{1}{x}, x \neq 0$ is shown on the right.



7. Let $g(x) = f(x) - x$. Since $y = f(x)$ and $y = x$ are continuous functions from $[a, b]$ to $[a, b]$, therefore $g(x)$ is continuous from $[-(b - a), b - a]$. Since the range of $f(x)$ assumes values of $[a, b]$, $f(a) \ge a$, $f(b) \le b$. If $f(a) = a$ or $f(b) = b$, the result follows easily by putting $x_0 = a$ or $x_0 = b$. So we assumes that $f(a) > a$ and $f(b) < b$.

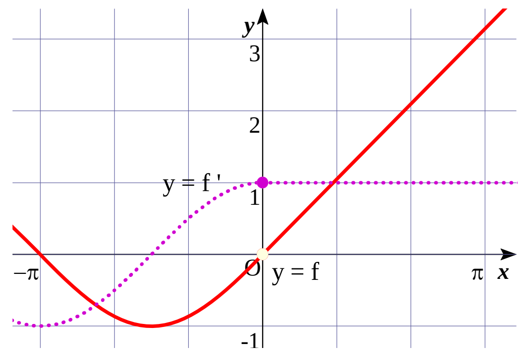
$\therefore$ $g(a) = f(a) - a > 0$ and $g(x_2) = f(b) - b < 0$ and there is a sign change.

$\therefore$ $\exists x_0$ s.t. $g(x_0) = f(x_0) - x_0 = 0$ or $f(x_0) = x_0$.

8. f is both continuous and differentiable at $x = 0$.

f is continuous since $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = f(0) = 0$

The graph on the right shows the situation.



Note that $f'(x) = \begin{cases} \cos x & \text{for } x \le 0 \\ 1 & \text{for } x > 0 \end{cases}$.

f is differentiable since $\lim\limits_{x\to 0^-} f'(x) = \lim\limits_{x\to 0^+} f'(x) = f'(0) = 1$.

9. If $x^2 > 1$, then as $n \to \infty$, $x^n \to \infty$. $\lim\limits_{n\to\infty} \dfrac{x^n f(x) + g(x)}{x^n + 1} = \lim\limits_{n\to\infty} \dfrac{f(x) + g(x)/x^n}{1 + 1/x^n} = \dfrac{f(x) + 0}{1 + 0} = f(x) = \phi(x)$.

If $x^2 < 1$, then as $n \to \infty$, $x^n \to 0$. $\lim\limits_{n\to\infty} \dfrac{x^n f(x) + g(x)}{x^n + 1} = \dfrac{0 + g(x)}{0 + 1} = g(x) = \phi(x)$.

$\therefore$ $\phi(x) = \lim\limits_{n\to\infty} \dfrac{x^n f(x) + g(x)}{x^n + 1}$, $x \ne 1$.

In general, $\phi(x)$ is not continuous on **R** since $\lim\limits_{x\to 1^+} \phi(x) = f(1)$, $\lim\limits_{x\to 1^-} \phi(x) = g(1)$, $\phi(1) = \dfrac{1}{2}[f(1) + g(1)]$

$\phi(x)$ is continuous on **R** if $f(1) = g(1)$ and $f(-1) = g(-1)$

10. Let $f(x) = \dfrac{1}{\sqrt{\cos x}}$, $f(x)$ is well defined $\Leftrightarrow$ $\cos x \ne 0$ $\Leftrightarrow$ $x \ne n\pi + \dfrac{\pi}{2}$, where $n \in \mathbb{N}$.

Domain of f(x) = $\mathbb{R} \setminus \left\{ n\pi + \dfrac{\pi}{2} \right\}$ and we need to prove that f(x) is piecewise continuous.

For any x in the domain, let h be any value such that $0 \le h \le \dfrac{\pi}{2}$.

$$\left| \frac{1}{\sqrt{\cos x}} - \frac{1}{\sqrt{\cos h}} \right| = \left| \frac{\sqrt{\cos h} - \sqrt{\cos x}}{\sqrt{\cos x}\sqrt{\cos h}} \right| = \left| \frac{\cos h - \cos x}{\left(\sqrt{\cos h} + \sqrt{\cos x}\right)\sqrt{\cos x}\sqrt{\cos h}} \right| = \left| \frac{2\sin\frac{x+h}{2}\sin\frac{x-h}{2}}{\left(\sqrt{\cos h} + \sqrt{\cos x}\right)\sqrt{\cos x}\sqrt{\cos h}} \right|$$

$$= \frac{2\left|\sin\frac{x+h}{2}\right|\left|\sin\frac{x-h}{2}\right|}{\left|\left(\sqrt{\cos h} + \sqrt{\cos x}\right)\sqrt{\cos x}\sqrt{\cos h}\right|} \le \frac{2 \times 1 \times \left|\frac{x-h}{2}\right|}{\left|\left(\sqrt{\cos h} + \sqrt{\cos x}\right)\sqrt{\cos x}\sqrt{\cos h}\right|} \to 0, \quad \text{as } x \to h.$$

$\therefore \quad \lim_{x\to h} f(x) = f(h)$ and $f(x)$ is continuous in its domain.

Let $g(x) = \sec x + \csc x = \dfrac{1}{\cos x} + \dfrac{1}{\sin x}$,

$f(x)$ is well defined $\Leftrightarrow$ $\cos x \neq 0$ and $\sin x \neq 0$ $\Leftrightarrow$ $x \neq \dfrac{n\pi}{2}$, where $n \in \mathbb{N}$.

Domain of $f(x) = \mathbb{R} \setminus \left\{\dfrac{n\pi}{2}\right\}$ and we need to prove that $f(x)$ is piecewise continuous.

For any x in the domain, let h be any value such that $0 \le h \le \dfrac{\pi}{2}$.

$$\left|\left(\frac{1}{\cos x}+\frac{1}{\sin x}\right)-\left(\frac{1}{\cos h}+\frac{1}{\sin h}\right)\right| \le \left|\frac{1}{\cos x}-\frac{1}{\cos h}\right| + \left|\frac{1}{\sin x}-\frac{1}{\sin h}\right| = \left|\frac{\cos h - \cos x}{\cos x \cos h}\right| + \left|\frac{\sin h - \sin x}{\sin h \sin x}\right|$$

$$= \left|\frac{2\sin\frac{x+h}{2}\sin\frac{x-h}{2}}{\cos x \cos h}\right| + \left|\frac{-2\cos\frac{x+h}{2}\sin\frac{x-h}{2}}{\sin h \sin x}\right| = \frac{2\left|\sin\frac{x+h}{2}\right|\left|\sin\frac{x-h}{2}\right|}{|\cos x \cos h|} + \frac{2\left|\cos\frac{x+h}{2}\right|\left|\sin\frac{x-h}{2}\right|}{|\sinh \sin x|}$$

$$\le \frac{2\times 1\times\left|\frac{x-h}{2}\right|}{|\cos x \cos h|} + \frac{2\times 1\times\left|\frac{x-h}{2}\right|}{|\sinh \sin x|} \to 0$$, as $x \to h$. $\therefore \quad \lim_{x\to h} f(x) = f(h)$ and $f(x)$ is cont. in its domain.

11. **(a)** $\lim_{x\to 0}\dfrac{x}{\sin x} = 1$, $f(0) = 1$. $\lim_{x\to 0}\dfrac{x}{\sin x} = f(0))$ $\quad\therefore$ $f(x)$ is continuous at $x = 0$.

(b) $f(x) = x - [x]$, $f(0) = 0$, $\lim_{x\to 0^-} f(x) = 0 - (-1) = 1$, $\lim_{x\to 0^+} f(x) = 0 - 0 = 0$

$f(1/2) = 1/2 - 0 = 1/2$, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 1/2$

$\therefore$ $f(x)$ is not continuous at $x = 0$, but it is continuous at $x = 1/2$.

(c) Take $x_n \in Q$, $x_n \to 0$, then $f(x_n) = 1$ and $\lim_{x_n\to 0} f(x) = 1$

Take $x_n \notin Q$, $x_n \to 0$, then $f(x_n) = 0$ and $\lim_{x_n\to 0} f(x) = 0$. Also, $f(0) = 1$

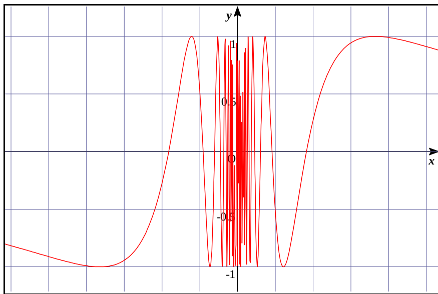
$\therefore$ $f(x)$ is not continuous at $x = 0$.

Take $x_n \in Q$, $x_n \to \pi$, then $f(x_n) = 1$ and $\lim_{x_n\to 0} f(x) = 1$

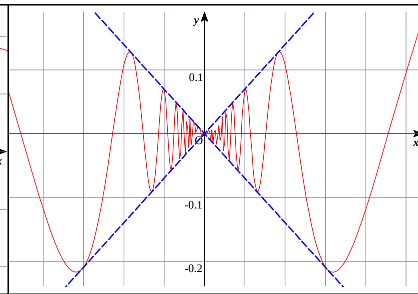
Take $x_n \notin Q$, $x_n \to \pi$, then $f(x_n) = 0$ and $\lim_{x_n\to 0} f(x) = 0$. Also, $f(\pi) = 0$

$\therefore$ $f(x)$ is also not continuous at $x = \pi$.

12. (a)

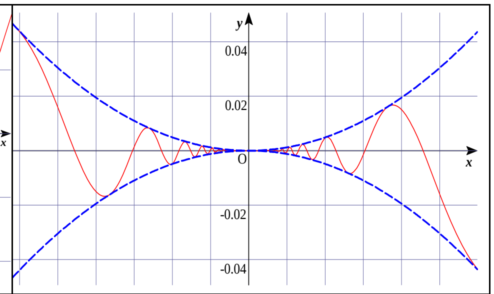


$y = f_0(x)$



$y = f_1(x)$

Two blue lines : $y = \pm x$



$y = f_2(x)$

Two blue lines : $y = \pm x^2$

(b) When $x = \dfrac{1}{\left(2m+\frac{1}{2}\right)\pi}$, $m \in \mathbb{Z}$. $y = 1$. When $x = \dfrac{1}{\left(2m-\frac{1}{2}\right)\pi}$, $m \in \mathbb{Z}$. $y = -1$.

Also, $f_0(0) = 0$. $\therefore$ f_0 is not continuous at $x = 0$

$$\left|x\sin\frac{1}{x}\right| = |x|\left|\sin\frac{1}{x}\right| \le |x| \Rightarrow \lim_{x\to 0}\left|x\sin\frac{1}{x}\right| \le \lim_{x\to 0}|x| = 0 \Rightarrow \lim_{x\to 0} x\sin\frac{1}{x} = 0 \Rightarrow \lim_{x\to 0} f_1(x) = 0 = f_1(0),$$

$$\left|x^2\sin\frac{1}{x}\right| = |x^2|\left|\sin\frac{1}{x}\right| \le |x^2| \Rightarrow \lim_{x\to 0}\left|x^2\sin\frac{1}{x}\right| \le \lim_{x\to 0}|x^2| = 0 \Rightarrow \lim_{x\to 0} x^2\sin\frac{1}{x} = 0 \Rightarrow \lim_{x\to 0} f_1(x) = 0 = f_1(0)$$

$\therefore$ f_1 and f_2 are continuous at $x = 0$.

(c) $$f_2'(0) = \lim_{h\to 0}\frac{f_2(0+h) - f_2(0)}{h} = \lim_{h\to 0}\frac{h^2\sin\frac{1}{h} - 0}{h} = \lim_{h\to 0}\left(h\sin\frac{1}{h}\right) = 0$$

When $x \ne 0$, $f_2'(x) = \dfrac{d}{dx}\left(x^2\sin\dfrac{1}{x}\right) = 2x\sin\dfrac{1}{x} - \cos\dfrac{1}{x}$ $\therefore$ $f_2'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & \text{, when } x \ne 0 \\ 0 & \text{, when } x = 0 \end{cases}$

(d) Since $\lim\limits_{x\to 0}\cos\dfrac{1}{x}$ does not exist, $f_2'(x)$ is not continuous at $x = 0$.

13. (a) $$\left|x^3\sin\frac{1}{x}\right| = |x^3|\left|\sin\frac{1}{x}\right| \le |x^3| \Rightarrow \lim_{x\to 0}\left|x^3\sin\frac{1}{x}\right| \le \lim_{x\to 0}|x^3| = 0 \Rightarrow \lim_{x\to 0} x^3\sin\frac{1}{x} = 0 \Rightarrow \lim_{x\to 0} f(x) = 0 = f(0)$$

$\therefore$ $f(x)$ is continuous at $x = 0$

(b) $$f'(0) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0}\frac{h^3\sin\frac{1}{h} - 0}{h} = \lim_{h\to 0}\left(h^2\sin\frac{1}{h}\right) = 0$$

When $x \ne 0$, $f'(x) = \dfrac{d}{dx}\left(x^3\sin\dfrac{1}{x}\right) = 3x^2\sin\dfrac{1}{x} - x\cos\dfrac{1}{x}$ $\therefore$

$$f'(x) = \begin{cases} 3x^2\sin\frac{1}{x} - x\cos\frac{1}{x} & \text{, when } x \ne 0 \\ 0 & \text{, when } x = 0 \end{cases}$$

(c) $\lim\limits_{x\to 0} f'(x) = \lim\limits_{x\to 0}\left(3x^2\sin\dfrac{1}{x} - x\cos\dfrac{1}{x}\right) = 0 = f(0),$ since $\sin\dfrac{1}{x}$ and $\cos\dfrac{1}{x}$ are bounded.

$\therefore$ $f'(x)$ is continuous at $x = 0$.